Stroppel

categorifying BMW

knot invariants	algebra	
Jones	TL alg.	
Homfly	Hecke alg	
Kauffman	BMW	Birman-Murakami-Wenzl

① Hecke alg. H_n for S_n over $\mathbb{Z}[v,v^{-1}]$

generators H_i $\quad 1 \leq i \leq n-1$

relations $H_i^2 = H_e + (v^{-1}-v) H_i$

$H_i H_j = H_j H_i \quad |i-j|>1$

$H_i H_j H_i = H_j H_i H_j \quad |i-j|=1$

"better" generators $C_i = H_i + v$

$C_i^2 = (v+v^{-1}) C_i$

$C_i C_j = C_j C_i \quad |i-j|>1$

$C_i C_j C_i + C_j = C_j C_i C_j + C_i \quad |i-j|=1$

KL basis $\{C_w\}_{w\in S_n}$ contains $C_{s_i} = C_i$

Soergel's categorification

$R = \mathbb{C}[X_1, X_2, \cdots, X_n] \quad \deg X_i = 2$

consider R-bimodules

$R_e = R$

$B_i = R \otimes_{R^{s_i}} R\langle -1\rangle$ $\quad 1 \leq i \leq n-1$ $\leftarrow$ invariants w.r.t. $s_i = (X_i X_{i+1})$

Th (Soergel)

These bimodules satisfy the Hecke relation with

$1 \leftrightarrow B_e$, $C_i \leftrightarrow B_i$ compositions $\leftrightarrow \otimes_R$

$v \leftrightarrow$ grading

$\mathcal{S}_R :=$ additive category generated by B_e, all B_i
closed under tensor product, finite direct sums
direct summands, grading shifts

$\left\{\begin{array}{l}\text{split} \\ \text{Grothendieck ring of } \mathcal{S}_R \cong H_n \\ \text{Indecomposable bimodules} \leftrightarrow \text{KL basis} \\ [A]\cdot[B] = [A \otimes B]\end{array}\right.$

e.g. $B_i \otimes_R B_i = R \otimes_{R^{s_i}} R \otimes_R R \otimes_{R^{s_i}} R\textcolor{red}{\langle -2 \rangle} \cong R \otimes_{R^{s_i}} R \otimes_{R^{s_i}} R\textcolor{red}{\langle -2 \rangle}$

$\left(R \cong R^{s_i} \oplus R^{s_i}\langle 2 \rangle \right.$ as R^{s_i} module $\left.\right)$

$\cong R \otimes_{R^{s_i}} R^{\textcolor{red}{\langle -2 \rangle}} \oplus R \otimes_{R^{s_i}} R \cancel{\langle 2 \rangle}$

$\cong B_i\textcolor{red}{\langle -1 \rangle} \oplus B_i \cancel{\langle 2 \rangle}\textcolor{red}{\langle 1 \rangle}$ $\textcolor{red}{= (v + v^{-1}) B_i}$

$|i-j|=1$

$B_i \otimes_R B_j \otimes_R B_i \cong B_i \oplus F$ for some F

$B_j \otimes_R B_i \otimes_R B_j \cong B_j \oplus F$

Braid groups

 $\leftrightarrow$ $H_i = C_i - v$

complex

$$R\langle 1\rangle \xrightarrow{\Delta} R\otimes_{R^{s_i}} R\langle -1\rangle$$

$$1 \mapsto X_i\otimes 1 + 1\otimes X_i$$

 $\leftrightarrow$ $H_i^{-1} = C_i - v^{-1}$

$$R\otimes_{R^{s_i}} R\langle -1\rangle \xrightarrow{\text{mult.}} R\langle -1\rangle$$

Th (Rouquier, S)

K_0(homotopy category of R-bimodules gen. by the B_e, B_i, , + closed under) $\cong$ Hecke alg.

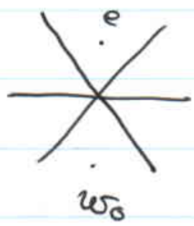
- get an interpret. of ,
- disadvantage : $H_i^2 = H_e + (v^{-1}-v)H_i$ only in K_0
- used for triply graded [KR]

Connection to $\mathcal{O}$

$\mathcal{O}(\mathfrak{gl}_n)_0$: principal block

$\cup$

$P(x)$ indec. proj. module $x \in S_n$



Th (Soergel)

$$\mathrm{End}_{\mathfrak{g}}(P(w_0)) \cong \mathbb{C}[X_1,\dots,X_n] \big/ \mathbb{C}[X_1,\dots,X_n]^{S_n}_+ \quad (\cong H^*(\mathcal{F}lag_n))$$

$/\!/\mathbb{C}$

most complicated one

cf. singular block $\to$ partial flag

Now $\mathbb{V} = \mathrm{Hom}_{\mathfrak{g}}(P(w_0), \cdot\,)$

$$\begin{tikzcd}
\mathcal{O}(\mathfrak{gl}_n)_0 \arrow[r, "\mathbb{V}"] \arrow[d, "F"', "\text{exact functor}"] & \operatorname{mod} C \arrow[d, dotted, "\overline{F}\ \text{exact}"] \\
\mathcal{O}(\mathfrak{gl}_n)_0 \arrow[r, "\mathbb{V}"] & \operatorname{mod} C
\end{tikzcd}$$

$P(w_0)$: char. by proj. & inj.

<u>Th.</u> Groth. ring

$$\left\{ \begin{array}{l} \text{of proj. functors} \\ \mathcal{O}(\mathfrak{gl}_n)_0 \to \mathcal{O}(\mathfrak{gl}_n)_0 \end{array} \right\} \overset{\mathbb{V}}{\longleftrightarrow} \left\{ \begin{array}{l} \text{Groth.} \\ \text{ring} \\ \text{of } \mathcal{S}_C \end{array} \right\} = \{ \mathcal{S}_R^{Gr.} \} \cong H_n$$

graded version

<u>NB.</u> true in the level of functors

TL alg. relation $C_i C_j C_i = C_i \quad |i-j| = 1$

$B_i \otimes_R B_j \otimes_R B_i \cong B_i \oplus F$

idea consider $\mathcal{S}_C / \underbrace{\langle \text{all } F\text{'s} \rangle} =: \mathcal{S}_C'$

これを実現する！ → parabolic!

<u>Th</u> Groth ring of

$$\left\{ \begin{array}{l} \text{proj. functors} \\ \bigoplus_{i=0}^{n} \mathcal{O}^{\mathfrak{p}_i}(\mathfrak{gl}_n)_0 \to \bigoplus \mathcal{O}^{\mathfrak{p}_i}(\mathfrak{gl}_n)_0 \end{array} \right\} \cong TL_{n,v}$$

parabolic w.r.t i

categorifies $U_q(\mathfrak{sl}_2) \circlearrowright V^{\otimes n} \circlearrowleft TL_{q,n}$

BMW

Kauffman 2-variable Laurent polynomial

regular invariant for links

(except Reidemeister I)

$$L_{\bigcirc} = 1 \quad , \quad L_{\gamma} = aL \quad , \quad L_{\gamma} = a^{-1}L$$

$$L_{\times} - L_{\times} = z\left(L_{\asymp} + L_{)(}\right)$$

L : regular $\Longrightarrow$ $S(K) = a^{-w(K)} L_K$

this is inv.

BMW alg. / $\mathbb{Z}[z^{\pm}, a^{\pm}]$ $\qquad \tau = 1 - \dfrac{a - a^{-1}}{z}$

generators $\quad g_i, g_i^{-1}, e_i \quad 1 \leq i \leq n-1$

relations $\quad$ ① braid relations for g_i

② TL relation for e_i, $\quad e_i^2 = \tau e_i$, $\quad e_i e_j = e_j e_i$ $\quad |i-j| > 1$

$e_i e_j e_i = e_i \quad |i-j| = 1$

③ delooping relation $\quad e_i g_i = g_i e_i = a e_i$

$e_i g_j e_i = a^{-1} e_i$

$e_i g_j g_i = g_j g_i e_j = e_i e_j \quad |i-j| = 1$

$g_i = \times$

$g_i^{-1} = \times$

$e_i = \asymp$

④ $g_i - g_i^{-1} = z(1 - e_i)$

deformation of Brauer algebra

basis : matching (with crossing $2n$ points)

e.g. $n=2$ $\quad ||, \asymp, \times$

$\dim BMW(n) = (n+1)!!$

$= 1 \cdot 3 \cdot 5 \cdots (2n+1)$

categorify !

1) Naive categorification

$\mathcal{N}$ = homotopy categorification of cpx generated by

$\mathcal{F}_c^!$ and $X[-\frac{1}{2}]\langle\frac{1}{2}\rangle$ where X as in Soergel's picture

TL e_i braid g_i

Th.

$$K_0(\mathcal{N}) \cong \widetilde{BMW}(n)$$

additive cat. or split Groth.

generated by all basis elements of BMW(n) without relation (4)

$$a^{-1} \leftrightarrow [\tfrac{1}{2}]\langle\tfrac{3}{2}\rangle$$
$$a \leftrightarrow [-\tfrac{1}{2}]\langle-\tfrac{3}{2}\rangle \qquad \tau \leftrightarrow \langle 1\rangle \oplus \langle -1\rangle$$

Rem. can be extended to BMW by considering complex of objects in $\mathcal{N}$

2) less naive

V = nat. rep. for $\mathcal{U}_q(\mathfrak{gl}_k)$

$$\mathcal{U}_q(\mathfrak{gl}_k) \curvearrowright V^{\otimes n} \curvearrowleft H_n \qquad \text{(A)}$$

restrict to $sp(k)$ if k even

$$\mathcal{U}_q(sp(k)) \curvearrowright V^{\otimes n} \curvearrowleft BMW(n) \quad \text{faithful if } k \gg 0$$

(B)

Categorify this !

Ⓐ Take $D^b\left(\bigoplus_{\lambda} \mathcal{O}^{p}_{\lambda}(\mathfrak{gl}_n)\right)$

$\lambda\ p$ runs all wts of $\mathfrak{gl}_n$ (dominant int.) which are from $\{0,\dots,k-1\}^n \subset \mathbb{Z}^n$

ex. $k=2$ $\{0,1\}$-seq. of n

$k=3$, $n=2$:

$(2,2)$
$(2,1)$ — orbit $(1\ 2)$
$(2,0)$ $\quad(0\ 2)$
$(1,1)$
$(1,0)$ $\quad(0\ 1)$
$(0,0)$

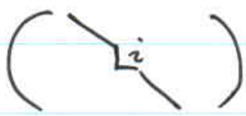
$U_q(\mathfrak{gl}_k)$-action

tensoring with natural rep. + its rep.

restricting to correct blocks

proj. 〃 〃

H_n-action $\quad V+q_i$ go to parabolic



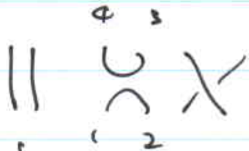
$\quad = C_i$

↑ need to go to D^b $\quad$ and back

How to do BMW?

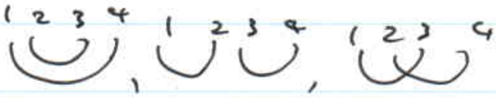
cf. Benkart–Ram

need action of e_i $\quad n=2$



↓

unfold



↓

permutation $\begin{pmatrix}1&2&3&4\\ 1&3&4&2\end{pmatrix} \in S_{2n}$

left cup points $\quad$ right ends but reversed

cf. 1 2 3 4

$\begin{pmatrix}1&2&3&4\\ 1&2&3&4\end{pmatrix}$

action of  $= e_i$ on $V\otimes V$ is

$$V\otimes V = \mathbb{C}\otimes(V\otimes V) \to (V\otimes V)\otimes(V^*\otimes V^*)\otimes V\otimes V$$

$$\xrightarrow{V^*\cong V} V\otimes V\otimes V\otimes V\otimes V\otimes V$$

braid $\otimes$ id

$$V\otimes V\otimes V\otimes V\otimes V\otimes V$$

$$\to V\otimes V\otimes(V^*\otimes V^*\otimes V\otimes V)$$

$$\to V\otimes V$$

categorify

choose 

cf.